Representation Theory Exercise Sheet 2

Alice Pozzi, Alex Torzewski

MATH0073

These questions cover roughly the first two weeks of lectures and are grouped by the relevant sections. The four questions with a boxed numbers, e.g. $\boxed{8.}$, are to be handed in during the lecture on Friday week 5 (14/02/2020). This excludes any starred parts, which are non-assessed.

1 Section 5

1. Find the Artin–Wedderburn decomposition of $\mathbb{C}[G]$ when G is D_8 (the dihedral group of order 8), Q_8 and A_4.

$\boxed{2.}$ Let ρ be a representation of G over k.

(a) Let $g \in Z(G)$. Show that $\rho(g)$ itself defines an element of $\mathrm{End}_G(\rho)$.

(b) Using Schur's lemma, show that if k is algebraically closed, ρ is irreducible and $g \in Z(G)$, then $\rho(g)$ must be a scalar.

(c) Deduce that if G is abelian and k is algebraically closed, then all irreducible representations are one-dimensional.

(d) Give a counter-example to the last statement holding if $k = \mathbb{R}$.

$\boxed{3.}$ Let H_3 be the group

$$H_3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{F}_3 \right\} \subset \mathrm{GL}_3(\mathbb{F}_3).$$

(a) Find the centre of H_3. Find H_3^{ab} and calculate the number of one dimensional representations of H_3.

(b) Calculate the number of irreducible representations of H_3 and their dimensions. Find the Artin–Wedderburn decomposition of $\mathbb{C}[H_3]$.

4. Let H_3 be as in Question 3. Let $\rho\colon H_3 \to \mathrm{GL}_3(\mathbb{F}_3)$ be the tautological representation given by considering elements of H_3 as matrices. Show that ρ is not semisimple.

5. (a) Prove that the upper triangular matrices whose diagonal entries are all 1 form a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ (Hint: you might want to look up the order of $\mathrm{GL}_n(\mathbb{F}_p)$).

   (b) Prove that every representation of a p-group (a group of prime power order) $\rho\colon G \to \mathrm{GL}_n(\mathbb{F}_p)$ is semisimple if and only if it is a direct sum of copies of the trivial representation $\mathbb{1}$.

2 Section 6+7

1. Let $\rho\colon G \to \mathrm{GL}_n(k)$ be a representation. If $\{e_1, ..., e_n\}$ denotes the standard basis of k^n, write $\{e_1^\vee, ..., e_n^\vee\}$ for the basis of $\mathrm{Hom}(k^n, k)$ given by $e_1^\vee(e_j) = \delta_{i,j}$. Prove that in this basis $\rho^\vee(g) = \rho(g^{-1})^t$. Deduce that $\chi_{V^\vee} = \overline{\chi}$.

2. Calculate the character table of $C_2 \times C_2$.

3. Calculate the conjugacy classes of Q_8.

4. Show that D_8 is not isomorphic to Q_8. Calculate the character table of D_8. Deduce that a group is not determined by its character table.

5. Show that the class function ψ of S_3:

	e	τ	σ
ψ	5	0	1

cannot be the character of a $\mathbb{C}$-representation of S_3.

6. There is a representation ρ_0 of Q_8 with character χ:

	1	-1	i	j	ij
χ	4	0	0	-2	0

Decompose ρ_0 as a sum of irreducible representations.

7. For a character χ, let
$$\ker \chi := \{g \in G \mid \chi(g) = \chi(1)\}.$$
Suppose that χ is the character of ρ. Show that $\ker \chi = \ker \rho$ (Hint: use Lemma 2.14). In particular, the kernel of a character is a subgroup.

8. A representation is called *faithful* if it is injective as a group homomorphism. Which irreducible characters of D_8 are faithful? Find a group with no faithful irreducible representations.

9. Let $g, g \in G$. Prove that $\chi(g) = \chi(g')$ for all characters χ if and only if g and g' are conjugate.

10. Prove that g is conjugate to g^{-1} if and only if for all characters χ we have that $\chi(g) \in \mathbb{R}$.

11. Let G be a finite group.
    (a) Given a group G, denote by $\hat{G}$ the following set:
    $$\hat{G} = \{\chi \mid \chi\colon G \to \mathbb{C}^\times \text{is a group homomorphism}\}.$$
    Show that $\hat{G}$ is an abelian group for the operation
    $$(\chi \cdot \psi)(g) := \chi(g) \cdot \psi(g)$$
    for $\chi, \psi \in \hat{G}$.
    (b) Show that $|G| = |\hat{G}| \iff G$ is abelian.
    (c) Show that if G is abelian, then G is canonically isomorphic to $\hat{\hat{G}}$.

12. (*Orthogonality of 1-dimensional characters*) Let G be a finite group, and let $\chi, \psi \in \hat{G}$.

(a) Show that

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)=\begin{cases}1, \text{if } \chi=1,\\ 0, \text{if } \chi\neq 1.\end{cases}$$

(Hint: Show that the sum is invariant under multiplication by $\chi(g)$ for $g\in G$. What does this tell you about $\chi(g)$?)

(b) Let $(\chi,\psi):=\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)}$. Deduce that

$$(\chi,\psi)=\begin{cases}1, \text{if } \chi=\psi,\\ 0, \text{if } \chi\neq \psi.\end{cases}$$